MATC44 Week 5 Notes

Examples of Principle of Invariants:

1. Let n be an odd number. The nums $1, 2, \dots, 2n$ are written on a board. The following steps are applied at each state:

a) Select 2 nums, say x and y.
b) Delete x and y.
c) Write $|x-y|$ on the board.

Is it possible that 2 is the final num?

Soln:
Let Q be the sum of the nums written on the board at each step. Suppose at step i we have the nums $a_i, \dots, a_j, x, y$ and suppose $x > y$. Then,

$Q_i = a_i + \dots + a_j + x + y$

Then, at step $i+1$, we have

$$\begin{aligned} Q_{i+1} &= a_i + \dots + a_j + x - y \\ &= a_i + \dots + a_j + x + y - 2y \\ &= Q_i - 2y \end{aligned}$$

This means that Q changes by an even number at each step.

$$\begin{aligned} Q_{initial} &= \frac{(2n)(2n+1)}{2} \\ &= \underbrace{(n)}_{odd}\underbrace{(2n+1)}_{odd} \end{aligned}$$

Therefore, $Q_{initial}$ is an odd num.
Since Q changes by an even num at each step, Q can never be an even num and can never be 2.

2. There are 100 0's (zeroes) and 100 1's on a board. At each state, we apply the following steps:

a) Select 2 nums, say x and y.
b) Delete x and y.
c) If $x=y$, write 0 on the board and if $x \neq y$, write 1 on the board.

Can 1 be the final num?

Soln:
Let Q be the sum of all the nums on the board.
$Q_{initial} = 100$, an even number.
Consider the following scenarios:

a) x and y are both 0. In this case, Q is unchanged.
b) x and y are both 1. In this case, Q decreases by 2.
c) $x=0$ and $y=1$ or $x=1$ and $y=0$. In this case, Q remains unchanged.

Therefore, Q changes by 0 or 2 at each state. As a result, because $Q_{initial}$ is even, Q can never be odd and can never be 1.

3. We have 3 tubes containing 8, 9, and 10 coins respectively. At each state, we apply the following steps:

a) Select 2 of the 3 tubes.

b) Remove 1 coin from each of the tubes we selected.

c) Add the 2 coins to the third tube.

Is it possible that one tube contains all the coins?

Soln:

Let A be the tube that initially has 8 coins.

Let B be the tube that initially has 9 coins.

Let C be the tube that initially has 10 coins.

Let Q_{ij} be the difference of the num of coins in tubes i and j.

Q_{AB} initial $= -1 \longrightarrow Q_{BA}$ initial $= 1$

Q_{AC} initial $= -2 \longrightarrow Q_{CA}$ initial $= 2$

Q_{BC} initial $= -1 \longrightarrow Q_{CB}$ initial $= 1$

Suppose at step i, there are x coins in A, y coins in B and z coins in C.

Suppose that we remove 1 coin from A and B and put it in C.

Consider the chart below:

Step	Q_A	Q_B	Q_C	$\lvert Q_{AB}\rvert$	$\lvert Q_{AC}\rvert$	$\lvert Q_{BC}\rvert$
i	x	y	z	$\lvert x-y\rvert$	$\lvert x-z\rvert$	$\lvert y-z\rvert$
$i+1$	$x-1$	$y-1$	$z+2$	$\lvert(x-1)-(y-1)\rvert$	$\lvert(x-1)-(z+2)\rvert$	$\lvert(y-1)-(z+2)\rvert$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\underbrace{|x-y|}_{\text{No Change}} \qquad \underbrace{|x-z-3| \quad |y-z-3|}_{\text{Change by 3 coins}}$$

Therefore, at each step, the difference of the num of coins in any 2 tubes is unchanged or changed by 3 or -3. If we are able to put all the coins in 1 tube, say C, then we have:

a) $|Q_{AB}| = 0$

b) $|Q_{AC}| = 27$ Note: $8+9+10=27$

c) $|Q_{BC}| = 27$

However, recall that:

a) $|Q_{AB} \text{ initial}| = 1$

b) $|Q_{AC} \text{ initial}| = 2$

c) $|Q_{BC} \text{ initial}| = 1$

Therefore, it is impossible to transfer all the coins into 1 tube.